

Existence of Nontrivial Solutions for p -Laplacian Equations in \mathbf{R}^N

Chungen Liu *

Youquan Zheng †

Abstract In this paper, we consider a p -Laplacian equation in \mathbf{R}^N with sign-changing potential and subcritical p -superlinear nonlinearity. By using the cohomological linking method for cones developed by Degiovanni and Lancelotti in 2007, an existence result is obtained. We also give a result on the existence of periodic solutions for one-dimensional p -Laplacian equations which can be proved by the same method.

Key words p -Laplacian equation; sign-changing potential; cohomological link; Cerami condition; periodic solution

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1 Introduction and main results

We mainly consider the following p -Laplacian equation in the entire space

$$\begin{cases} -\Delta_p u + U(x)|u|^{p-2}u = f(x, u), \\ u \in W^{1,p}(\mathbf{R}^N, \mathbf{R}), \end{cases} \quad (1.1)$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplacian operator with $p > 1$.

For $p = 2$, (1.1) turns into a kind of Schrödinger equation of the form

$$-\Delta u + U(x)u = f(x, u), \quad u \in H^1(\mathbf{R}^N, \mathbf{R}), \quad (1.2)$$

which has been studied extensively. In [4, 5, 6, 25], (1.2) with a constant sign potential $U(x)$ was considered. More precisely, the potential in these papers is of the form $a_0(x) + \lambda a(x)$,

*School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China.

Partially supported by NFS of China and 973 Program of STM.

E-mail: liucg@nankai.edu.cn

†School of Mathematical Sciences, Nankai University, Tianjin 300071, China.

E-mail: zhengyq@mail.nankai.edu.cn

$a_0(x)$ has positive lower bound, $a(x) \geq 0$ and $\lambda > 0$ large enough. And in [12, 13, 19], the authors considered (1.2) with a potential $U(x)$ that may change sign.

For general $p > 1$, most of the work, as the authors of this paper known, deal with the problem (1.1) with a constant sign potential $U(x)$, see for example [14, 26, 23] and the reference therein.

In this paper, we consider (1.1) with sign-changing potential and subcritical p -superlinear nonlinearity, moreover, periodic conditions on the potential and nonlinearity are not needed. Assume that $U(x)$ is of the form $b(x) - \lambda V(x)$, here λ is a real number, $b(x)$, $V(x)$ and $f(x, t)$ satisfy the following conditions

$$(B) \quad b \in C(\mathbf{R}^N, \mathbf{R}), \quad \inf_{x \in \mathbf{R}^N} b(x) \geq b_0 > 0, \quad meas(\{x \in \mathbf{R}^N : b(x) \leq M\}) < \infty, \quad \forall M > 0,$$

$$(V) \quad V \in L^\infty(\mathbf{R}^N, \mathbf{R}),$$

$$(f_1) \quad f \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R}), \quad \exists q \in (p, p^*) \text{ s.t. } |f(x, t)| \leq C(1 + |t|^{q-1}), \quad f(x, t)t \geq 0, \quad \forall t \in \mathbf{R},$$

$$(f_2) \quad \lim_{|t| \rightarrow \infty} \frac{f(x, t)t}{|t|^p} = +\infty \text{ uniformly in } x \in \mathbf{R}^N,$$

$$(f_3) \quad f(x, t) = o(|t|^{p-1}) \text{ as } |t| \rightarrow 0, \text{ uniformly in } x \in \mathbf{R}^N,$$

$$(f_4) \quad \exists \theta \geq 1 \text{ s.t. } \theta \mathcal{F}(x, t) \geq \mathcal{F}(x, st), \quad \forall (x, t) \in \mathbf{R}^N \times \mathbf{R} \text{ and } s \in [0, 1],$$

here we have set $F(x, t) = \int_0^t f(x, t)dt$, $\mathcal{F}(x, t) = f(x, t)t - pF(x, t)$, $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = +\infty$ if $p \geq N$ and $meas(\cdot)$ means the Lebesgue measure in \mathbf{R}^N .

Our main result reads as

Theorem 1.1 *If (B), (V) and (f₁)-(f₄) hold, the problem (1.1) possesses a nontrivial solution for every $\lambda \in \mathbf{R}$.*

We remark that the condition $\inf_{x \in \mathbf{R}^N} b(x) \geq b_0 > 0$ is not essential, it can be replaced by the condition $\inf_{x \in \mathbf{R}^N} b(x) > -\infty$. First note that the case $\lambda = 0$ can be replaced by the case $V = 0$ with a nonzero λ , we can always assume that $\lambda \neq 0$. If $\inf_{x \in \mathbf{R}^N} b(x) > -c_0$ for some $c_0 > 0$, one can replace b and V by $b + c_0$ and $V + \frac{c_0}{\lambda}$, then $b + c_0$ and $V + \frac{c_0}{\lambda}$ satisfy conditions (B) and (V). For $p = 2$, the condition (f₄) was introduced in [21], and for $p \neq 2$ it was introduced in [27]. Condition (B) was first introduced in [5], and then was used by many authors, for example, [33].

When dealing with superlinear problem, one usually needs a growth condition together with the following classical condition which was introduced by Ambrosetti and Rabinowitz

in [1],

$$\text{There exists } \mu > 2 \text{ such that for } u \neq 0 \text{ and } x \in \mathbf{R}^N, 0 < \mu F(x, u) \leq u f(x, u). \quad (1.3)$$

Since then, many authors tried to weaken this condition, see [13, 18, 26, 27, 28, 30, 32].

In [30] the authors obtained a weak solution of (1.2) under the following conditions

$$(C_1) \quad U(x) \in C(\mathbf{R}^N, \mathbf{R}), \quad \inf_{x \in \mathbf{R}^N} U(x) \geq U_0 > 0, \quad U(x) \text{ is 1-periodic in each of } x_i, \quad i = 1, \dots, N,$$

$$(C_2) \quad f(x, t) \in C^1 \text{ is 1-periodic in each of } x_i, \quad i = 1, \dots, N, \quad f'_t \text{ is a Caratheodory function and there exists } C > 0, \text{ such that } |f'_t(x, t)| \leq C(1 + |t|^{2^*-2}), \quad \lim_{|t| \rightarrow \infty} \frac{|f(x, t)|}{|t|^{2^*-1}} = 0, \text{ uniformly in } x \in \mathbf{R}^N,$$

$$(C_3) \quad f(x, t) = o(|t|), \text{ as } |t| \rightarrow 0, \text{ uniformly in } x,$$

$$(C_4) \quad \lim_{|u| \rightarrow \infty} \frac{F(x, u)}{u^2} = \infty, \text{ uniformly in } x,$$

$$(C_5) \quad \frac{f(x, t)}{|t|} \text{ is strictly increasing in } t.$$

And in [26] the author got a weak solution of (1.1) with the following assumptions

$$(D_1) \quad V \in C(\mathbf{R}^N), \text{ is 1-periodic in } x_i, \quad i = 1, \dots, N \text{ and } 0 < \alpha \leq V(x) \leq \beta < +\infty,$$

$$(D_2) \quad f \in C(\mathbf{R}^N \times \mathbf{R}) \text{ is 1-periodic in } x_i, \quad i = 1, \dots, N, \text{ and } \lim_{|t| \rightarrow \infty} \frac{f(x, t)}{|t|^{p^*-1}} = 0,$$

$$(D_3) \quad \lim_{|t| \rightarrow \infty} \frac{F(x, t)}{|t|^p} = +\infty \text{ uniformly in } x \in \mathbf{R}^N,$$

$$(D_4) \quad f(x, t) = o(|t|^{p-2}t) \text{ as } |t| \rightarrow 0, \text{ uniformly in } x \in \mathbf{R}^N,$$

$$(D_5) \quad \text{There exists } \theta \geq 1 \text{ such that } \theta \mathcal{F}(x, t) \geq \mathcal{F}(x, st) \text{ for } (x, t) \in \mathbf{R}^N \times \mathbf{R} \text{ and } s \in [0, 1].$$

From above we can see (1.3) is weaken to (C₄) with the cost (C₅) and to (D₃) with the cost (D₅), respectively. And condition (D₅) is weaker than (C₅)(c.f.[27]). In our result, (f₂)takes place the condition (1.3) but we need (f₄).

We should also mention that there is another line to weaken (1.3). In [13] for λ large enough the authors got a nontrivial solution of (1.2) with $U(x) = \lambda V(x)$ under the following conditions

$$(E_1) \quad V \in C(\mathbf{R}^N, \mathbf{R}), \quad V \text{ is bounded below, } V^{-1}(0) \text{ has nonempty interior,}$$

(E₂) there exists $M > 0$ such that the set $\{x \in \mathbf{R}^N | V(x) < M\}$ is nonempty and has finite measure,

(E₃) $f \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$, $F(x, u) \geq 0$ for all (x, u) , $f(x, u) = o(u)$ uniformly in x as $u \rightarrow 0$,

(E₄) $F(x, u)/u^2 \rightarrow \infty$ uniformly in x as $|u| \rightarrow \infty$,

(E₅) $\frac{1}{2}f(x, u)u - F(x, u) > 0$ whenever $u \neq 0$,

(E₆) $|f(x, u)|^\tau \leq a_1 \left(\frac{1}{2}f(x, u)u - F(x, u) \right) |u|^\tau$ for some $a_1 > 0$, $\tau > \max\{1, N/2\}$ and all (x, u) with $|u|$ large enough.

The authors of [13] also proved that conditions (E₄)(E₅)(E₆) are weaker than (1.3).

The main method used in the proof of Theorem 1.1 is the linking structure over cones which was developed in [9]. We will use the Cerami condition instead of (PS) condition. This method is also valid in finding periodic solutions for one-dimensional p -Laplacian equation. We will give a brief argument in section 5 for this topic.

The paper is organized as follows. In section 2, we give the variational settings, recall a critical point theorem and some important properties of cohomological index. In section 3, an eigenvalue problem is studied. We get a divergent sequence of eigenvalues for this eigenvalue problem by cohomological index theory. In section 4, we give a proof of Theorem 1.1. In section 5, we state an existence result for the periodic solutions of one-dimensional p -Laplacian equation.

2 Preliminaries

Let $\mathcal{W} := \{u \in W^{1,p}(\mathbf{R}^N, \mathbf{R}) : \int_{\mathbf{R}^N} (|\nabla u|^p + b(x)|u|^p) dx < \infty\}$ with $b(x)$ satisfying the condition (B). Then \mathcal{W} is a reflexive, separable Banach space with norm $\|u\| = (\int_{\mathbf{R}^N} (|\nabla u|^p + b(x)|u|^p) dx)^{\frac{1}{p}}$. From Gagliardo-Nirenberg inequality and Hölder inequality, we have $\mathcal{W} \hookrightarrow L^q(\mathbf{R}^N, \mathbf{R})$ for $p \leq q \leq p^*$. Moreover, we have the following compactness result. It was proved in [34] in the case $p = 2$. For the general case, the proof is similar. We give it here for reader's convenience.

Lemma 2.1 $\mathcal{W} \hookrightarrow L^s(\mathbf{R}^N, \mathbf{R})$ for $p \leq s < p^*$.

Proof: Let $\{u_n\} \subset \mathcal{W}$ be a bounded sequence of \mathcal{W} such that $u_n \rightharpoonup u$ weakly in \mathcal{W} . Then, by the Sobolev embedding theorem, $u_n \rightarrow u$ strongly in $L^s_{loc}(\mathbf{R}^N, \mathbf{R})$ for $p \leq s < p^*$. We first claim that

$$u_n \rightarrow u \text{ strongly in } L^p(\mathbf{R}^N, \mathbf{R}). \quad (2.4)$$

In fact, by the uniformly convex properties of $L^p(\mathbf{R}^N, \mathbf{R})$, we only need to prove that $\alpha_n := \|u_n\|_p \rightarrow \|u\|_p$ (cf. p295 in [11]). Assume, up to subsequence, that $\alpha_n \rightarrow \alpha$.

Set

$$\begin{aligned} B_R &= \{x \in \mathbf{R}^N : |x| < R\}, \\ A(R, M) &= \{x \in \mathbf{R}^N \setminus B_R : b(x) \geq M\}, \\ B(R, M) &= \{x \in \mathbf{R}^N \setminus B_R : b(x) < M\}. \end{aligned}$$

Then

$$\int_{A(R, M)} |u_n|^p dx \leq \int_{\mathbf{R}^N} \frac{b(x)}{M} |u_n|^p dx \leq \frac{\|u_n\|_p^p}{M}.$$

Choose $t \in (1, \frac{p^*}{p})$ and t' such that $\frac{1}{t} + \frac{1}{t'} = 1$. Then

$$\int_{B(R, M)} |u_n|^p dx \leq \left(\int_{B(R, M)} |u_n|^{pt} dx \right)^{\frac{1}{t}} (meas(B(R, M)))^{\frac{1}{t'}} \leq C \|u_n\|^p (meas(B(R, M)))^{\frac{1}{t'}}.$$

Since $\{\|u_n\|\}$ is bounded and condition (B) holds, we may choose R, M large enough such that $\frac{\|u_n\|_p^p}{M}$ and $meas(B(R, M))$ are small enough. Hence, $\forall \varepsilon > 0$, we have

$$\int_{\mathbf{R}^N \setminus B_R} |u_n|^p dx = \int_{A(R, M)} |u_n|^p dx + \int_{B(R, M)} |u_n|^p dx < \varepsilon.$$

Thus,

$$\begin{aligned} \|u\|_p^p &= \|u\|_{L^p(B_R)}^p + \|u\|_{L^p(\mathbf{R}^N \setminus B_R)}^p \\ &\geq \lim_{n \rightarrow \infty} \|u_n\|_{L^p(B_R)}^p = \lim_{n \rightarrow \infty} (\|u_n\|^p - \|u_n\|_{L^p(\mathbf{R}^N \setminus B_R)}^p) \geq \alpha^p - \varepsilon. \end{aligned}$$

On the other hand, let Ω be an arbitrary domain in \mathbf{R}^N , then

$$\int_{\Omega} |u_n|^p dx \leq \int_{\mathbf{R}^N} |u_n|^p dx \rightarrow \alpha^p,$$

hence $\|u\|_p \leq \alpha$. Thanks to the arbitrariness of ε , we have $\alpha = \|u\|_p$. So (2.4) is proved.

Finally, it is easy to prove that $u_n \rightarrow u$ in $L^s(\mathbf{R}^N, \mathbf{R})$ for $p \leq s < p^*$. In fact, if $s \in (p, p^*)$, there is a number $\lambda \in (0, 1)$ such that $\frac{1}{s} = \frac{\lambda}{p} + \frac{1-\lambda}{p^*}$. Then by the Hölder inequality,

$$\|u_n - u\|_s^s = \int_{\mathbf{R}^N} |u_n - u|^{\lambda s} |u_n - u|^{(1-\lambda)s} dx \leq \|u_n - u\|_p^{\lambda s} \|u_n - u\|_{p^*}^{(1-\lambda)s}.$$

Since u_n is bounded in $L^{p^*}(\mathbf{R}^N, \mathbf{R})$ and $\|u_n - u\|_p \rightarrow 0$, we have $u_n \rightarrow u$ in $L^s(\mathbf{R}^N, \mathbf{R})$. ■

In the following, we consider the C^1 functional $\Phi : \mathcal{W} \rightarrow \mathbf{R}$ defined by

$$\Phi(u) = \frac{1}{p} \int_{\mathbf{R}^N} (|\nabla u|^p + b(x)|u|^p) dx - \frac{\lambda}{p} \int_{\mathbf{R}^N} V(x)|u|^p dx - \int_{\mathbf{R}^N} F(x, u) dx. \quad (2.5)$$

It is clear that critical points of Φ are weak solutions of (1.1). In order to find a critical point of this functional, we will use the following critical point theorem. It was proved in [9], where the functional was supposed to satisfy the (PS) condition. Recently, in [8], the author extended it to more general case (the functional space is completely regular topological space or metric space). If the functional space is a real Banach space, according to the proof of Theorem 6.10 in [8], the Cerami condition is sufficient for the compactness of the set of critical points at a fixed level and the first deformation lemma to hold (see [31]). So this critical point theorem still hold under the Cerami condition.

Lemma 2.2 ([9]) *Let \mathcal{W} be a real Banach space and let C_- , C_+ be two symmetric cones in \mathcal{W} such that C_+ is closed in \mathcal{W} , $C_- \cap C_+ = \{0\}$ and*

$$i(C_- \setminus \{0\}) = i(\mathcal{W} \setminus C_+) = m < \infty.$$

Define the following four sets by

$$D_- = \{u \in C_- : \|u\| \leq r_-\},$$

$$S_+ = \{u \in C_+ : \|u\| = r_+\},$$

$$Q = \{u + te : u \in C_-, t \geq 0, \|u + te\| \leq r_-\}, \quad e \in \mathcal{W} \setminus C_-,$$

$$H = \{u + te : u \in C_-, t \geq 0, \|u + te\| = r_-\}.$$

Then $(Q, D_- \cup H)$ links S_+ cohomologically in dimension $m + 1$ over \mathbf{Z}_2 . Moreover, suppose $\Phi \in C^1(\mathcal{W}, \mathbf{R})$ satisfying the Cerami condition, and $\sup_{x \in D_- \cup H} \Phi(x) < \inf_{x \in S^+} \Phi(x)$, $\sup_{x \in Q} \Phi(x) < \infty$. Then Φ has a critical value $d \geq \inf_{x \in S^+} \Phi(x)$.

For convenience, let us recall the definition and some properties of the cohomological index of Fadell-Rabinowitz for a \mathbf{Z}_2 -set, see [16, 17, 31] for details. For simplicity, we only consider the usual \mathbf{Z}_2 -action on a linear space, i.e., $\mathbf{Z}_2 = \{1, -1\}$ and the action is the usual multiplication. In this case, the \mathbf{Z}_2 -set A is a symmetric set with $-A = A$.

Let E be a normed linear space. We denote by $\mathcal{S}(E)$ the set of all symmetric subsets of E which do not contain the origin of E . For $A \in \mathcal{S}(E)$, denote $\bar{A} = A/\mathbf{Z}_2$. Let $\rho : \bar{A} \rightarrow \mathbf{R}P^\infty$ be the classifying map and $\rho^* : H^*(\mathbf{R}P^\infty) = \mathbf{Z}_2[\omega] \rightarrow H^*(\bar{A})$ the induced homomorphism of the cohomology rings. The cohomological index of A , denoted by $i(A)$, is defined by $\sup\{k \geq 1 : \rho^*(\omega^{k-1}) \neq 0\}$. We list some properties of the cohomological index here for further use in this paper. Let $A, B \in \mathcal{S}(E)$, there hold

- (i1) (**monotonicity**) if $A \subseteq B$, then $i(A) \leq i(B)$,

- (i2) (**invariance**) if $h : A \rightarrow B$ is an odd homeomorphism, then $i(A) = i(B)$,
- (i3) (**continuity**) if C is a closed symmetric subset of A , then there exists a closed symmetric neighborhood N of C in A , such that $i(N) = i(C)$, hence the interior of N in A is also a neighborhood of C in A and $i(\text{int}N) = i(C)$,
- (i4) (**neighborhood of zero**) if V is bounded closed symmetric neighborhood of the origin in E , then $i(\partial V) = \dim E$.

3 Eigenvalue problem

In this section, we consider the following eigenvalue problem

$$\begin{cases} -\Delta_p u + b(x)|u|^{p-2}u = \lambda V(x)|u|^{p-2}u, \\ u \in W^{1,p}(\mathbf{R}^N, \mathbf{R}). \end{cases}$$

We assume V satisfies condition (V) and further assume that $V^+(x) := \frac{V(x)+|V(x)|}{2} \neq 0$ on some positive measure subset of \mathbf{R}^N in this section. Define on \mathcal{W} the functionals

$$H(u) = \frac{1}{p} \int_{\mathbf{R}^N} (|\nabla u|^p + b(x)|u|^p) dx,$$

$$I(u) = \frac{1}{p} \int_{\mathbf{R}^N} V(x)|u|^p dx.$$

Then

$$H \in C^1(\mathcal{W}, \mathbf{R}), \quad \langle H'(u), v \rangle = \int_{\mathbf{R}^N} (|\nabla u|^{p-2} \nabla u \nabla v + b(x)|u|^{p-2} uv) dx$$

and

$$I \in C^1(\mathcal{W}, \mathbf{R}), \quad \langle I'(u), v \rangle = \int_{\mathbf{R}^N} V(x)|u|^{p-2} uv dx.$$

Our aim is to solve the eigenvalue problem

$$H'(u) = \lambda I'(u). \tag{3.6}$$

Lemma 3.1 *For any $u, v \in \mathcal{W}$, it holds that*

$$\langle H'(u) - H'(v), u - v \rangle \geq (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|).$$

Proof: We follow the idea of the proof of Lemma 2.3 in [24]. By direct computations, we have

$$\begin{aligned}\langle H'(u) - H'(v), u - v \rangle &= \int_{\mathbf{R}^N} |\nabla u|^p + |\nabla v|^p - |\nabla u|^{p-2} \nabla u \cdot \nabla v - |\nabla v|^{p-2} \nabla v \cdot \nabla u \, dx \\ &\quad + \int_{\mathbf{R}^N} b(x) (|u|^p + |v|^p - |u|^{p-2} uv - |v|^{p-2} vu) \, dx.\end{aligned}$$

From the definition of the norm in \mathcal{W} , we can get

$$\begin{aligned}\langle H'(u) - H'(v), u - v \rangle &= \|u\|^p + \|v\|^p - \int_{\mathbf{R}^N} (|\nabla u|^{p-2} \nabla u \cdot \nabla v + b(x) |u|^{p-2} uv) \, dx \\ &\quad - \int_{\mathbf{R}^N} (|\nabla v|^{p-2} \nabla v \cdot \nabla u + b(x) |v|^{p-2} vu) \, dx.\end{aligned}$$

Applying Hölder inequality,

$$\begin{aligned}&\int_{\mathbf{R}^N} (|\nabla u|^{p-2} \nabla u \cdot \nabla v + b(x) |u|^{p-2} uv) \, dx \\ &\leq \left(\int_{\mathbf{R}^N} |\nabla u|^p \, dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbf{R}^N} |\nabla v|^p \, dx \right)^{\frac{1}{p}} + \left(\int_{\mathbf{R}^N} b(x) |u|^p \, dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbf{R}^N} b(x) |v|^p \, dx \right)^{\frac{1}{p}}.\end{aligned}$$

Using the following inequality

$$(a + b)^\alpha (c + d)^{1-\alpha} \geq a^\alpha c^{1-\alpha} + b^\alpha d^{1-\alpha}$$

which holds for any $\alpha \in (0, 1)$ and for any $a > 0, b > 0, c > 0, d > 0$, set $\alpha = \frac{p-1}{p}$ and

$$a = \int_{\mathbf{R}^N} |\nabla u|^p \, dx, \quad b = \int_{\mathbf{R}^N} b(x) |u|^p \, dx, \quad c = \int_{\mathbf{R}^N} |\nabla v|^p \, dx, \quad d = \int_{\mathbf{R}^N} b(x) |v|^p \, dx,$$

we can deduce that

$$\int_{\mathbf{R}^N} (|\nabla u|^{p-2} \nabla u \cdot \nabla v + b(x) |u|^{p-2} uv) \, dx \leq \|u\|^{p-1} \|v\|.$$

Similarly, we can obtain

$$\int_{\mathbf{R}^N} (|\nabla v|^{p-2} \nabla v \cdot \nabla u + b(x) |v|^{p-2} vu) \, dx \leq \|v\|^{p-1} \|u\|.$$

Therefore, we have

$$\begin{aligned}\langle H'(u) - H'(v), u - v \rangle &\geq \|u\|^p + \|v\|^p - \|u\|^{p-1} \|v\| - \|v\|^{p-1} \|u\| \\ &= (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|).\end{aligned}$$

■

Lemma 3.2 *If $u_n \rightharpoonup u$ and $\langle H'(u_n), u_n - u \rangle \rightarrow 0$, then $u_n \rightarrow u$ in \mathcal{W} .*

Proof: Since \mathcal{W} is a reflexive Banach space, it is isometrically isomorphic to a locally uniformly convex space, so as it was proved in [11], weak convergence and norm convergence imply strong convergence. Therefore we only need to show that $\|u_n\| \rightarrow \|u\|$.

Note that

$$\lim_{n \rightarrow \infty} \langle H'(u_n) - H'(u), u_n - u \rangle = \lim_{n \rightarrow \infty} (\langle H'(u_n), u_n - u \rangle - \langle H'(u), u_n - u \rangle) = 0.$$

By Lemma 3.1 we have

$$\langle H'(u_n) - H'(u), u_n - u \rangle \geq (\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|) \geq 0.$$

Hence $\|u_n\| \rightarrow \|u\|$ as $n \rightarrow \infty$ and the assertion follows. \blacksquare

Lemma 3.3 *I' is weak-to-strong continuous, i.e. $u_n \rightharpoonup u$ in \mathcal{W} implies $I'(u_n) \rightarrow I'(u)$.*

Proof: This is a direct consequence of Theorem 1.22 in [34] and Lemma 2.1. \blacksquare

Lemma 3.4 *If $u_n \rightharpoonup u$ in \mathcal{W} , then $I(u_n) \rightarrow I(u)$.*

Proof:

$$\begin{aligned} p|I(u_n) - I(u)| &= |\langle I'(u_n), u_n \rangle - \langle I'(u), u \rangle| \\ &= |\langle I'(u_n) - I'(u), u_n \rangle + \langle I'(u), u_n - u \rangle| \\ &\leq \|I'(u_n) - I'(u)\| \|u_n\| + o(1). \end{aligned}$$

Because $u_n \rightharpoonup u$, u_n is bounded. From Lemma 3.3, we have $I(u_n) \rightarrow I(u)$. \blacksquare

Set $\mathcal{M} = \{u \in \mathcal{W} : I(u) = 1\}$. Clearly, $I(u) = \frac{1}{p} \langle I'(u), u \rangle$, so 1 is a regular value of the functional I . Hence by the implicit theorem, \mathcal{M} is a C^1 -Finsler manifold. It is complete, symmetric, since I is continuous and even. Moreover, 0 is not contained in \mathcal{M} , so the trivial \mathbf{Z}_2 -action on \mathcal{M} is free. Set $\tilde{H} = H|_{\mathcal{M}}$.

Lemma 3.5 *If $u \in \mathcal{M}$ satisfies $\tilde{H}(u) = \lambda$ and $\tilde{H}'(u) = 0$, then (λ, u) is a solution of (3.6).*

Proof: By Proposition 3.14.9 in [31], the norm of $\tilde{H}'(u) \in T_u^* \mathcal{M}$ is given by $\|\tilde{H}'(u)\|_u^* = \min_{\mu \in \mathbf{R}} \|H'(u) - \mu I'(u)\|^*$ (here the norm $\|\cdot\|_u^*$ is the norm in the fibre $T_u^* \mathcal{M}$, and $\|\cdot\|^*$ is the operator norm, the minimal can be attained was proved in Lemma 3.14.10 in [31]). Hence there exists $\mu \in \mathbf{R}$ such that $H'(u) - \mu I'(u) = 0$, that is (μ, u) is a solution of (3.6) and $\lambda = \tilde{H}(u) = \frac{1}{p} \langle H'(u), u \rangle = \frac{1}{p} \langle \mu I'(u), u \rangle = \mu \frac{1}{p} \langle I'(u), u \rangle = \mu I(u) = \mu$. \blacksquare

Lemma 3.6 *\tilde{H} satisfies the (PS) condition, i.e. if (u_n) is a sequence on \mathcal{M} such that $\tilde{H}(u_n) \rightarrow c$, and $\tilde{H}'(u_n) \rightarrow 0$, then up to a subsequence $u_n \rightarrow u \in \mathcal{M}$ in \mathcal{W} .*

Proof: First, from the definition of H , we can deduce that (u_n) is bounded. Then, up to a subsequence, u_n converges weakly to some u , by Lemma 3.4, we have $I(u) = 1$, so $u \in \mathcal{M}$.

From $\tilde{H}'(u_n) \rightarrow 0$, we have $H'(u_n) - \mu_n I'(u_n) \rightarrow 0$ for a sequence of real numbers (μ_n) . So $\langle H'(u_n) - \mu_n I'(u_n), u_n \rangle \rightarrow 0$, thus we get $\mu_n \rightarrow c$. By Lemma 3.3, we have $H'(u_n) \rightarrow cI'(u)$. Hence

$$\langle H'(u_n), u_n - u \rangle = \langle H'(u_n) - cI'(u), u_n - u \rangle + \langle cI'(u), u_n - u \rangle \rightarrow 0.$$

By Lemma 3.2, we obtain $u_n \rightarrow u$. ■

Let \mathcal{F} denote the class of symmetric subsets of \mathcal{M} , $\mathcal{F}_n = \{M \in \mathcal{F} : i(M) \geq n\}$ and

$$\lambda_n = \inf_{M \in \mathcal{F}_n} \sup_{u \in M} \tilde{H}(u). \quad (3.7)$$

Since $\mathcal{F}_n \supset \mathcal{F}_{n+1}$, $\lambda_n \leq \lambda_{n+1}$.

Lemma 3.7 *For every \mathcal{F}_n , there is a symmetric compact set $M \in \mathcal{F}_n$.*

Proof: We follow the idea of the proof of Theorem 3.2 in [20]. Since $\text{meas}\{x \in \mathbf{R}^N : V(x) > 0\} > 0$, it implies that $\forall n \in \mathbf{N}^*$, there exist n open balls $(B_i)_{1 \leq i \leq n}$ in \mathbf{R}^N such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\text{meas}(\{x \in \mathbf{R}^N : V(x) > 0\} \cap B_i) > 0$. Approximating the characteristic function $\chi_{\{x \in \mathbf{R}^N : V(x) > 0\} \cap B_i}$ by $C^\infty(\mathbf{R}^N, \mathbf{R})$ functions in $L^p(\mathbf{R}^N, \mathbf{R})$, we can infer that there exists a sequence $\{u_i\}_{1 \leq i \leq n} \subseteq C^\infty(\mathbf{R}^N, \mathbf{R})$ such that $\int_{\mathbf{R}^N} V(x)|u_i|^p dx > 0$ for all $i = 1, \dots, n$ and $\text{supp } u_i \cap \text{supp } u_j = \emptyset$ when $i \neq j$. Normalizing u_i , we assume that $I(u_i) = 1$. Denote U_n the space spanned by $(u_i)_{1 \leq i \leq n}$. $\forall u \in U_n$, we have $u = \sum_{i=1}^n \alpha_i u_i$ and $I(u) = \sum_{i=1}^n |\alpha_i|^p$. So $u \rightarrow \left(I(u)\right)^{\frac{1}{p}}$ defines a norm on U_n . Since U_n is n dimensional, this norm is equivalent to $\|\cdot\|$. Thus $\{u \in U_n : I(u) = 1\} \subseteq \mathcal{M}$ is compact with respect to the norm $\|\cdot\|$ and by (i4), $i(\{u \in U_n : I(u) = 1\}) = n$. So $\{u \in U_n : I(u) = 1\} \in \mathcal{F}_n$. ■

By Lemma 3.7, we have $\lambda_n < +\infty$, and by condition (B), there holds $\lambda_n \geq 0$. Furthermore, by Lemma 3.6 and Proposition 3.14.7 in [31], we see that λ_n is sequence of critical values of \tilde{H} and $\lambda_n \rightarrow +\infty$, as $n \rightarrow \infty$. By Lemma 3.5 we get a divergent sequence of eigenvalues for problem (3.6). So we have the following result.

Theorem 3.8 *Problem (3.6) has an increasing sequence eigenvalues λ_n which are defined by (3.7) and $\lambda_n \rightarrow +\infty$, as $n \rightarrow \infty$.*

Lemma 3.9 *Set*

$$\mu_n = \inf_{K \in \mathcal{F}_n^c} \sup_{u \in K} H(u), \quad (3.8)$$

where $\mathcal{F}_n^c = \{K \in \mathcal{F}_n : K \text{ is compact}\}$. Then we have $\lambda_n = \mu_n$.

Proof: From Lemma 3.7, $\mathcal{F}_n^c \neq \emptyset$ and so $\mu_n < +\infty$. It is obvious that $\lambda_n \leq \mu_n$. If $\lambda_n < \mu_n$, there is $M \in \mathcal{F}_n$ such that $\sup_{u \in M} H(u) < \mu_n$. The closure \overline{M} of M in \mathcal{M} is still in \mathcal{F}_n , by continuity of H , $\sup_{u \in \overline{M}} H(u) < \mu_n$ holds. Applying the property (i3) of the cohomological index, we can find a small open neighborhood $A \in \mathcal{F}_n$ of \overline{M} in \mathcal{M} such that $\sup_{u \in A} H(u) < \mu_n$. As it was proved in the proof of Proposition 3.1 in [9], for every symmetric open subset A of \mathcal{M} , there holds $i(A) = \sup\{i(K) : K \text{ is compact and symmetric with } K \subseteq A\}$. So we can choose a symmetric compact subset $K \subseteq A$ with $i(K) \geq n$ and $\sup_{u \in K} H(u) < \mu_n$. This contradicts to the definition of μ_n . Therefore we have $\lambda_n = \mu_n$. \blacksquare

Motivated by Theorem 3.2 in [9], we have the following statement.

Theorem 3.10 *If $\lambda_m < \lambda_{m+1}$ for some $m \in \mathbf{N}^*$, then*

$$i(\{u \in \mathcal{W} \setminus \{0\} : H(u) \leq \lambda_m I(u)\}) = i(\{u \in \mathcal{W} : H(u) < \lambda_{m+1} I(u)\}) = m.$$

Proof: Suppose $\lambda_m < \lambda_{m+1}$. If we set $A = \{u \in \mathcal{M} : H(u) \leq \lambda_m\}$ and $B = \{u \in \mathcal{M} : H(u) < \lambda_{m+1}\}$, by the definition (3.7), we have $i(A) \leq m$. Assume that $i(A) \leq m-1$. Thanks to (i3), there exists a symmetric neighborhood \mathcal{N} of A in \mathcal{M} satisfying $i(\mathcal{N}) = i(A)$. By the equivariant deformation theorem(see [7]), there exists $\delta > 0$ and an odd continuous map $\iota : \{u \in \mathcal{M} : H(u) \leq \lambda_m + \delta\} \rightarrow \{u \in \mathcal{M} : H(u) \leq \lambda_m - \delta\} \cup \mathcal{N} = \mathcal{N}$. Hence $i(\{u \in \mathcal{M} : H(u) \leq \lambda_m + \delta\}) \leq m-1$. By (3.7), there exists $M \in \mathcal{F}_m$ such that $\sup_{u \in M} H(u) < \lambda_m + \delta$. So $M \subseteq \{u \in \mathcal{M} : H(u) \leq \lambda_m + \delta\}$ and thus $i(M) \leq m-1$. This contradicts to the fact that $M \in \mathcal{F}_m$. Thus we have $i(A) = m$. By the invariance of the cohomological index under odd homeomorphism and the functionals H, I are p -homogeneous, we have $i(\{u \in \mathcal{W} \setminus \{0\} : H(u) \leq \lambda_m I(u)\}) = m$.

Since $A \subseteq B$ and $i(A) = m$, we have $i(B) \geq m$. Assume that $i(B) \geq m+1$. As in the proof of Lemma 3.9, there exists a symmetric, compact subset K of B with $i(K) \geq m+1$. Since $\max_{u \in K} H(u) < \lambda_{m+1} = \mu_{m+1}$, this contradicts to definition (3.8). By the invariance of the cohomological index under odd homeomorphism and the functionals H, I are p -homogeneous, we have $i(\{u \in \mathcal{W} : H(u) < \lambda_{m+1} I(u)\}) = m$. \blacksquare

4 Proof of the main theorem

Set $J(u) = \int_{\mathbf{R}^N} F(x, u) dx$, by the definition of H and I in section 3, we can write the functional Φ defined in section 2 as

$$\Phi(u) = H(u) - \lambda I(u) - J(u), \quad u \in \mathcal{W}.$$

It follows from Lemma 1.22 in [34] that J' is compact.

Replacing (λ, V) with $(-\lambda, -V)$, we can assume that $\lambda \geq 0$.

First, we consider the case $V^+(x) \neq 0$ on some positive measure subset of \mathbf{R}^N and there exist $m \geq 1$ such that $\lambda_m \leq \lambda < \lambda_{m+1}$. Set

$$C_- = \{u \in \mathcal{W} : H(u) \leq \lambda_m I(u)\}, \quad (4.9)$$

$$C_+ = \{u \in \mathcal{W} : H(u) \geq \lambda_{m+1} I(u)\}. \quad (4.10)$$

It is easy to see that C_- , C_+ are two symmetric closed cones in \mathcal{W} and $C_- \cap C_+ = \{0\}$.

By Theorem 3.10 we have

$$i(C_- \setminus \{0\}) = i(\mathcal{W} \setminus C_+) = m. \quad (4.11)$$

Theorem 4.1 *There exist $r_+ > 0$ and $\alpha > 0$ such that $\Phi(u) > \alpha$ for $u \in C_+$ and $\|u\| = r_+$.*

Proof : Let $\varepsilon > 0$ be small enough, from (f_1) and (f_3) , we have $|F(x, t)| \leq \varepsilon|t|^p + C_\varepsilon|t|^q$, by the Sobolev embedding inequality, for $u \in C_+$, we can get

$$\begin{aligned} \Phi(u) &= H(u) - \lambda I(u) - J(u) \\ &= H(u) - \frac{\lambda}{\lambda_{m+1}} \lambda_{m+1} I(u) - J(u) \\ &\geq H(u) - \frac{\lambda}{\lambda_{m+1}} H(u) - \varepsilon \int_{\mathbf{R}^N} |u|^p dx - C_\varepsilon \int_{\mathbf{R}^N} |u|^q dx \\ &\geq H(u) - \frac{\lambda}{\lambda_{m+1}} H(u) - \frac{\varepsilon}{b_0} \int_{\mathbf{R}^N} b(x) |u|^p dx - C_\varepsilon \int_{\mathbf{R}^N} |u|^q dx \\ &\geq (1 - \frac{\lambda}{\lambda_{m+1}} - \frac{\varepsilon}{b_0}) H(u) - C_\varepsilon \int_{\mathbf{R}^N} |u|^q dx \\ &\geq \frac{(1 - \frac{\lambda}{\lambda_{m+1}} - \frac{\varepsilon}{b_0})}{p} \|u\|^p - C \|u\|^q. \end{aligned} \quad (4.12)$$

We remind that in the second inequality of (4.12), the condition (B) has been applied.

Since $p < q$, the assertion follows. ■

Since $\lambda \geq \lambda_m$, by (f_1) it holds that

$$\Phi(u) \leq 0, \quad \forall u \in C_-. \quad (4.13)$$

Set $\mathbf{R}^+ = [0, +\infty)$. Following the idea of the proof of Theorem 4.1 in [9], we have

Theorem 4.2 *Let $e \in \mathcal{W} \setminus C_-$, there exists $r_- > r_+$ such that $\Phi(u) \leq 0$ for $u \in C_- + \mathbf{R}^+ e$ and $\|u\| \geq r_-$.*

Proof : Define another norm on \mathcal{W} by $\|u\|_V := (\int_{\mathbf{R}^N} (|V(x)| + 1)|u|^p dx)^{1/p}$. Then the same reason as the proof of Theorem 4.1 in [9], there exists some constant $b > 0$ such that $\|u + te\| \leq b\|u + te\|_V$ for every $u \in C_-$, $t \geq 0$ and some $b > 0$. That is

$$\int_{\mathbf{R}^N} (|\nabla(u + te)|^p + b(x)|u + te|^p) dx \leq b^p \int_{\mathbf{R}^N} (|V(x)| + 1)|u + te|^p dx. \quad (4.14)$$

Let $\{u_k\}$ be a sequence such that $\|u_k\| \rightarrow +\infty$ and $u_k \in C_- + \mathbf{R}^+e$. Set $v_k = \frac{u_k}{\|u_k\|}$, then, up to a subsequence, $\{v_k\}$ converges to some v weakly in \mathcal{W} and a.e. in \mathbf{R}^N . Note that Lemma 3.4 is also true for functional $\int_{\mathbf{R}^N} (|V(x)| + 1)|u|^p dx$, $u \in \mathcal{W}$, it follows from (4.14) that $\int_{\mathbf{R}^N} (|V(x)| + 1)|v|^p dx \geq \frac{1}{b^p}$. So $|v| \neq 0$ on a positive measure set Ω_0 . Since $\lim_{|t| \rightarrow \infty} \frac{f(x, t)t}{|t|^p} = +\infty$ implies $\lim_{|t| \rightarrow \infty} \frac{F(x, t)}{|t|^p} = +\infty$, from (f₂) we have

$$\lim_{k \rightarrow \infty} \frac{F(x, u_k(x))}{\|u_k\|^p} = \lim_{k \rightarrow \infty} \frac{F(x, \|u_k\|v_k(x))}{\|u_k\|^p |v_k(x)|^p} |v_k(x)|^p = +\infty, \quad x \in \Omega_0.$$

By (f₁) and Fatou's lemma we can get

$$\frac{\int_{\mathbf{R}^N} F(x, u_k) dx}{\|u_k\|^p} \rightarrow +\infty, \quad \text{as } k \rightarrow \infty.$$

By the arbitrariness of the sequence $\{u_k\}$, we have $\frac{\int_{\mathbf{R}^N} F(x, u) dx}{\|u\|^p} \rightarrow +\infty$ as $\|u\| \rightarrow +\infty$ and $u \in C_- + \mathbf{R}^+e$. Noting that

$$\frac{\Phi(u)}{\|u\|^p} = \frac{1}{p} - \frac{\lambda I(u)}{\|u\|^p} - \frac{\int_{\mathbf{R}^N} F(x, u) dx}{\|u\|^p}$$

and by conditions (B) and (V),

$$\left| \frac{I(u)}{\|u\|^p} \right| \leq \frac{C \int_{\mathbf{R}^N} |u|^p dx}{\|u\|^p} \leq \frac{C \int_{\mathbf{R}^N} b(x)|u|^p dx}{\|u\|^p} \leq C,$$

the assertion follows. ■

Theorem 4.3 Φ satisfies the Cerami condition. i.e., for any sequence $\{u_k\}$ in \mathcal{W} satisfying $(1 + \|u_k\|)\Phi'(u_k) \rightarrow 0$ and $\Phi(u_k) \rightarrow c$ possesses a convergent subsequence.

Proof : Let $\{u_k\}$ be a sequence in \mathcal{W} satisfying $(1 + \|u_k\|)\Phi'(u_k) \rightarrow 0$ and $\Phi(u_k) \rightarrow c$. We claim that $\{u_k\}$ is bounded in \mathcal{W} . Otherwise, if $\|u_k\| \rightarrow \infty$, we consider $w_k := \frac{u_k}{\|u_k\|}$. Then, up to subsequence, we get $w_k \rightharpoonup w$ in \mathcal{W} , $w_k \rightarrow w$ in $L^s(\mathbf{R}^N)$ for $p \leq s < p^*$ and $w_k(x) \rightarrow w(x)$ a.e. $x \in \mathbf{R}^N$ as $k \rightarrow \infty$. If $w \neq 0$ in \mathcal{W} , since $\Phi'(u_k)u_k \rightarrow 0$, that is to say

$$\int_{\mathbf{R}^N} (|\nabla u_k|^p + b(x)|u_k|^p) dx - \lambda \int_{\mathbf{R}^N} V(x)|u_k|^p dx - \int_{\mathbf{R}^N} f(x, u_k)u_k dx \rightarrow 0, \quad (4.15)$$

from condition (V), we have $\frac{|\int_{\mathbf{R}^N} V(x)|u_k|^p dx|}{\|u_k\|^p} \leq C$, so by dividing the left hand side of (4.15) with $\|u_k\|^p$ there holds

$$\left| \int_{\mathbf{R}^N} \frac{f(x, u_k)u_k}{\|u_k\|^p} dx \right| \leq C. \quad (4.16)$$

On the other hand, by Fatou's lemma and condition (f₂) we have

$$\int_{\mathbf{R}^N} \frac{f(x, u_k)u_k}{\|u_k\|^p} dx = \int_{\{w_k \neq 0\}} |w_k|^p \frac{f(x, u_k)u_k}{|u_k|^p} dx \rightarrow \infty,$$

this contradicts to (4.16).

If $w = 0$ in \mathcal{W} , inspired by [21], we choose $t_k \in [0, 1]$ such that $\Phi(t_k u_k) := \max_{t \in [0, 1]} \Phi(t u_k)$. For any $\beta > 0$ and $\tilde{w}_k := (2p\beta)^{1/p} w_k$, by Lemma 3.4 and the compactness of J' we have that

$$\Phi(t_k u_k) \geq \Phi(\tilde{w}_k) = 2\beta - \lambda \int_{\mathbf{R}^N} V(x) |\tilde{w}_k|^p dx - \int_{\mathbf{R}^N} F(x, \tilde{w}_k) dx \geq \beta,$$

when k is large enough, this implies that

$$\lim_{k \rightarrow \infty} \Phi(t_k u_k) = \infty. \quad (4.17)$$

Since $\Phi(0) = 0$, $\Phi(u_k) \rightarrow c$, we have $t_k \in (0, 1)$. By the definition of t_k ,

$$\langle \Phi'(t_k u_k), t_k u_k \rangle = 0. \quad (4.18)$$

From (4.17), (4.18), we have

$$\Phi(t_k u_k) - \frac{1}{p} \langle \Phi'(t_k u_k), t_k u_k \rangle = \int_{\mathbf{R}^N} \left(\frac{1}{p} f(x, t_k u_k) t_k u_k - F(x, t_k u_k) \right) dx \rightarrow \infty.$$

By (f₄), there exists $\theta \geq 1$ such that

$$\int_{\mathbf{R}^N} \left(\frac{1}{p} f(x, u_k) u_k - F(x, u_k) \right) dx \geq \frac{1}{\theta} \int_{\mathbf{R}^N} \left(\frac{1}{p} f(x, t_k u_k) t_k u_k - F(x, t_k u_k) \right) dx \rightarrow \infty. \quad (4.19)$$

On the other hand,

$$\int_{\mathbf{R}^N} \left(\frac{1}{p} f(x, u_k) u_k - F(x, u_k) \right) dx = \Phi(u_k) - \frac{1}{p} \langle \Phi'(u_k), u_k \rangle \rightarrow c_0. \quad (4.20)$$

(4.19) and (4.20) are contradiction. Hence $\{u_k\}$ is bounded in \mathcal{W} . So up to a subsequence, we can assume that $u_k \rightharpoonup u$ for some $u \in \mathcal{W}$.

Since $\Phi'(u_k) = H'(u_k) - \lambda I'(u_k) - J'(u_k) \rightarrow 0$ and I', J' are compact, we have that $H'(u_k) \rightarrow \lambda I'(u) + J'(u)$ in \mathcal{W}^* . So

$$\langle H'(u_k), u_k - u \rangle = \langle H'(u_k) - (\lambda I'(u) + J'(u)), u_k - u \rangle + \langle \lambda I'(u) + J'(u), u_k - u \rangle \rightarrow 0.$$

By Lemma 3.2, $u_k \rightarrow u$ in \mathcal{W} . ■

Proof of Theorem 1.1 Define D_- , S_+ , Q , H as lemma 2.2, then from Theorem 4.1, $\Phi(u) \geq \alpha > 0$ for every $u \in S_+$, from Theorem 4.2, $\Phi(u) \leq 0$ for every $u \in D_- \cup H$ and Φ is bounded on Q . Applying Theorem 4.3, it follows that Φ has a critical value $d \geq \alpha > 0$. Hence u is a nontrivial weak solution of (1.1).

For the cases $0 \leq \lambda < \lambda_1$ or $V^+(x) \equiv 0$, set $C_- = \{0\}$ and $C_+ = \mathcal{W}$, it is easy to see that the arguments in this section are also valid. So we get a nontrivial solution and the proof of Theorem 1.1 is complete. ■

5 Periodic problem for one-dimensional p -Laplacian equation

In this section, we state a result which can be proved by the same methods as in the proof of Theorem 1.1. We only outline the main points. Our result reads as

Theorem 5.1 *If $p > 1$, $V \in L^\infty(\mathbf{R}, \mathbf{R})$, $f \in C(\mathbf{R} \times \mathbf{R}, \mathbf{R})$ satisfies (f_1) – (f_4) , both $V(t)$ and $f(t, u)$ are 1-periodic in t , then*

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda V(t)|u|^{p-2}u + f(t, u), \\ u(0) = u(1), u'(0) = u'(1). \end{cases} \quad (5.21)$$

has a nontrivial solution for every $\lambda \in \mathbf{R}$.

The periodic solution of p -Laplacian equation has been considered in many papers, for example, [2, 3, 29]. Up to the author's knowledge, Theorem 5.1 is new.

Let $\mathcal{W} := W^{1,p}(S^1, \mathbf{R})$ with the norm $\|u\| = (\int_{S^1} |\nabla u|^p + |u|^p dt)^{\frac{1}{p}}$, here $S^1 = \mathbf{R}/\mathbf{Z}$. Then \mathcal{W} is a reflexive, separable Banach space. And \mathcal{W} can be embedded into $L^q(\mathbf{R}, \mathbf{R})$ for any $p \leq q < \infty$. As in section 3, we consider the eigenvalue problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda V(t)|u|^{p-2}u, \\ u(0) = u(1), u'(0) = u'(1). \end{cases}$$

We can get a divergent sequence of eigenvalues defined by $\lambda_n = \inf_{M \in \mathcal{F}_n} \sup_{u \in M} \int_{S^1} |\nabla u|^p + |u|^p dt$ if $V^+(t) \neq 0$ on a positive measure subset of S^1 , here \mathcal{F}_n is the class of symmetrical subsets with Fadell-Rabinowitz index greater than n of $\mathcal{M} := \{u \in \mathcal{W} : \int_{S^1} |u|^p dt = 1\}$. And if $\lambda_m < \lambda_{m+1}$ for some $m \in \mathbf{N}^*$, then $i(\{u \in \mathcal{W} \setminus \{0\} : \int_{S^1} |\nabla u|^p + |u|^p dt \leq \lambda_m \int_{S^1} |u|^p dt\}) = i(\{u \in \mathcal{W} : \int_{S^1} |\nabla u|^p + |u|^p dt < \lambda_{m+1} \int_{S^1} |u|^p dt\}) = m$. Then arguing as in section 4,

consider the functional on \mathcal{W}

$$\Phi(u) = \frac{1}{p} \int_{S^1} (|\nabla u|^p + |u|^p) dt - \frac{\lambda}{p} \int_{S^1} V(t) |u|^p dt - \int_{S^1} F(t, u) dt,$$

assume $\lambda \geq 0$, $V^+(t) \neq 0$ on a positive measure subset of S^1 and there exists $m \in \mathbf{N}^*$ such that $\lambda_m \leq \lambda < \lambda_{m+1}$, Set

$$C_- = \{u \in \mathcal{W} : \int_{S^1} |\nabla u|^p + |u|^p dt \leq \lambda_m \int_{S^1} |u|^p dt\},$$

$$C_+ = \{u \in \mathcal{W} : \int_{S^1} |\nabla u|^p + |u|^p dt \geq \lambda_{m+1} \int_{S^1} |u|^p dt\},$$

then we have

- (1) There exist $r_+ > 0$ and $\alpha > 0$ such that $\Phi(u) > \alpha$ for $u \in C_+$ and $\|u\| = r_+$,
- (2) Let $e \in \mathcal{W} \setminus C_-$, there exists $r_- > r_+$ such that $\Phi(u) \leq 0$ for $u \in C_- + \mathbf{R}^+ e$ and $\|u\| \geq r_-$,
- (3) Φ satisfies the Cerami condition.

Then from Lemma 2.2, we can get a nontrivial solution for (5.21). The cases for $0 \leq \lambda < \lambda_1$ or $V^+(x) \equiv 0$ are similar as in the proof of Theorem 1.1.

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